Math 210B Lecture 12 Notes

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1 Tensor Products of Algebras and Homomorphism Groups

1.1 Tensor products of algebras

Let A, B, C be R-algebras, where R is a commutative ring. Let M and N be R-balanced A-B and B-C bimodules, respectively.

Definition 1.1. An **R-balanced** bimodule M is a module such that $rm = rm$ for all $r \in R, m \in M$.

This is equivalent to M being a $A \otimes_R B^{\text{op}}$-module. Then $M \otimes_B N$ becomes an R-balanced A-C bimodule:

$$a(m \otimes n) = am \otimes n, \qquad (m \otimes n)c = m \otimes nc.$$

We can also take tensor products of R-algebras, to get an R-algebra $A \otimes_R B$. We can define this by

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'.$$

Proposition 1.1. *Multiplication is well-defined.*

Proof. We want to construct $A \times B \to \operatorname{End}_R(A \otimes_R B)$ sending $(a, b) \mapsto \varphi_{a,b} = (a' \otimes b' \mapsto aa' \otimes bb')$. To show that $\varphi_{a,b}$ is well defined, we want a map $A \times B \to A \otimes_R B$ sending $(a', b') \mapsto aa' \otimes bb'$. By the universal property of the tensor product, we get a unique map $A \otimes_R B \to A \otimes_R B$, which we can set to be $\varphi_{a,b}$.

Now we want to show that our original map is bilinear. Check that

$$(ra_1 + a_2, b) \mapsto \varphi_{ra_1+a_2,b} = r\varphi_{a_1,b} + r\varphi_{a_2}.$$

By the universal property, we get a map $A \otimes_R B \to \operatorname{End}_R(A \otimes_R B)$ sending $a \otimes b \mapsto (a' \otimes b' \mapsto aa' \otimes bb')$. So then we get a map $A \otimes_R \times A \otimes_R B \to A \otimes_R B$ sending $(a \otimes b, a; \otimes b') \mapsto aa' \otimes bb'$. So the operation is well-defined. $\square$

Example 1.1. Let R be a commutative ring. Then $R[x] \otimes_R R[y] \cong R[x,y]$ by specifying $(x^i, y^j) \mapsto x^i y^j$ and extending this map to be bilinear. This map is surjective because we get every monomial in $R[x,y]$. Since $R[x,y]$ is free on the monomials $x^i y^j$, we can define an inverse map defined by $x^i y^j \mapsto x^i \otimes y^j$.

Example 1.2. Let G be a group. The R**-group ring** of G, $R[G]$, is the set of sums $\sum_{g \in G} a_g[g]$, where $a_g \in R$ and $a_g = 0$ for all but finitely many g. We can define multiplication on this by extending the multiplication on monomials defined by $[g] \cdot [h] = [gh]$.

1.2 Homomorphism groups

Example 1.3. Let M, N be R-modules. Then $\mathrm{Hom}_R(M,N)$ is an R-module: Let $\phi, \psi \in \mathrm{Hom}_R(M,N)$. Then we can define $(r\varphi)(m) := \varphi(rm) = r\varphi(m)$ and $(\varphi + \psi)(m) = \varphi(m) + \varphi(m)$. These are still R-module homomorphisms:

$$(r\varphi)(m)(sm) = \varphi(rsm) = \varphi(srm) = s\varphi(rm) = s(r\varphi)(m)$$

for $r, s \in R$.

Remark 1.1. If M, N are A-modules, then $\mathrm{Hom}_A(M,N)$ is an R-module but not an A-module.

Example 1.4. Let M be an R-balanced A-B bimodule, and let N be an R-balanced A-C bimodule. Then $\mathrm{Hom}_A(M,N)$ is a B-C bimodule by defining

$$(b\varphi)(m) := \varphi(mb), \qquad (\varphi c)(m) = \varphi(m)c.$$

Check that everything is balanced.

$$\mathrm{Hom}_A(\cdot,\cdot) : A \otimes_R B^{\mathrm{op}}\text{-mod} \to B \times A \otimes_R B^{\mathrm{op}}\text{-mod} \to B \otimes_R C^{\mathrm{op}}\text{-mod is a bifunctor.}$$

$$\mathrm{Hom}_A(M \prod_{i \in I} N_i) \cong \prod_{i \in I} \mathrm{Hom}_A(M, N_i).$$

$$\mathrm{Hom}_A(\bigoplus_{i \in I} M_i, N) \cong \prod_{i \in I} \mathrm{Hom}_A(M_i, N).$$

Definition 1.2. If F is a field, and V is an F vector space, we can define the **dual vector space**, $V^* = \mathrm{Hom}_F(V, F)$.

1.3 Dual vector spaces

If we have a map $f : V \to W$, we get a map $f^* : W^* \to V^*$ defined by $f^*(\varphi)(v) = \varphi \circ f(v)$, so $V \mapsto V^*$ is a contravariant functor from F-vector spaces to F-vector spaces.

If V has basis $v_1, \dots, v_n$, then there is a **dual basis** $\varphi_1, \dots, \varphi_n$ of V^* given by

$$\varphi_i(v_j) = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

So $V \cong V^*$ if V is finite dimensional. This is not the case if V is infinite-dimensional.

The functor $V \mapsto V^{**}$ covariant. We get $\Phi : V \to V^{**}$ given by $\Phi(v)(f) = f(v)$. Check that Φ is F-linear.

Proposition 1.2. $\Phi : V \to V^{**}$ *is injective.*

Proof. If $\Phi(v) = 0$, then $f(v) = 0$ for all $f \in V^*$; if $v \neq 0$, extend v to a basis B. Then there exists $f_v \in V^*$ such that $f_v(v) = 1$ and $f_v(w) = 0$ for all $w \in B$ with $w \neq v$. This is a contradiction. □

However, Φ is not always an isomorphism. If $V = \bigoplus_{i \in I}$, then $V = \mathrm{Hom}(\bigoplus_{i \in I} F, F) = \prod_{i \in I} \mathrm{Hom}(F, F) = \prod_{i \in I} F$, which is bigger than V. So V^{**} will be even bigger.

Proposition 1.3. *If W is finite dimensional over F, then* $\mathrm{Hom}_F(V, W) \cong V^* \otimes_F W$ *via* $f \otimes w \mapsto (v \mapsto f(v)w)$.

Proof. $W = \bigotimes_{i=1}^n F w_i$. Then

$$V^* \otimes_F \bigoplus_{i=1}^n F \cong \bigoplus_{i=1}^n V^* \otimes_F F \cong \bigoplus_{i=1}^n V^* \cong \bigoplus_{i=1}^n \mathrm{Hom}(V, F) \cong \mathrm{Hom}(V, \bigoplus_{i=1}^n F).$$

This isomorphism is precisely the map you get from composing these isomorphisms. □

1.4 Adjointness of Hom and ⊗

Theorem 1.1. *Let A, B, C be R-algebras, and let M, N, L be R-balanced A-B, B-C, and A-C bimodules, respectively. Then* $\mathrm{Hom}_A(M \otimes_B N, L) \cong \mathrm{Hom}_B(N, \mathrm{Hom}_A(M, L))$ *as right C-modules. Moreover, these are natural in M, N, L. In fact, we have* $t_M : B \otimes_R C^{\mathrm{op}}\text{-mod} \to A \otimes_R C^{\mathrm{op}}\text{-mod}$

$$\begin{array}{ccc} N & \longrightarrow & M \otimes_R N \\ \downarrow \lambda & & \downarrow \mathrm{id}_M \otimes_R \lambda \\ N' & \longrightarrow & M \otimes_R N' \end{array}$$

and $h_M : A \otimes_R C^{op}\text{-mod} \to B \otimes_R C^{op}\text{-mod}$ *such that* $\mathrm{Hom}_A(tM(N), L) \cong \mathrm{Hom}_B(N, h_M(L))$ *is natural in N and L; i.e. t_M is left adjoint to h_M.*

We will prove this next time.